FIXED POINT SETS OF HOMEOMORPHISMS OF COMPACT SURFACES*

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ABSTRACT

Every closed and non-empty subset of a compact surface S can be the fixed point set of a homeomorphism, and S also admits fixed point free homeomorphisms if it does not have the fixed point property. A partial extension to higher dimensions states that every closed and non-empty subset of a compact n-manifold can be the fixed point set of a surjective self-map.

1. Introduction

This paper contributes to the question whether any closed and non-empty subset A of a topological space X can be the fixed point set of a homeomorphism of X. It is known that this is the case if X is an even-dimensional ball, but that A has to satisfy additional conditions if X is an odd-dimensional ball or a dendrite [3, 4, 5].

Here we show that any closed and non-empty subset of a compact surface S can be the fixed point set of a homeomorphism (Theorem 1), and that there also exists a fixed point free homeomorphism if S does not have the fixed point property (Theorem 2). The proofs consist of elementary constructions which are related tot hose used in [3] and which do not generalize to higher dimensional manifolds.

Only for spheres of higher dimensions do the methods of [3] still yield homeomorphisms with prescribed closed fixed point sets (Theorem 3). For arbitrary compact n-manifolds we show, with the help of the concept of a path field [1, 2], that every closed and non-empty subset can at least be the fixed point set of a

* This research was partially supported by the National Research Council of Canada (Grant A 7579).

Received July 20, 1971

surjective self-map (Theorem 4). It would be interesting to know whether this result can be extended to homeomorphisms.

2. Homeomorphisms of compact surfaces with given non-empty fixed point sets

We show here that every closed and non-empty subset of a compact surface (i.e. compact and connected 2-manifold) can be the fixed point set of a homeomorphism. The proof uses a representation of the surface as a quotient space of a polygon and differs somewhat in the orientable and in the non-orientable case.

THEOREM 1. *Let S be a compact surface and A a closed and non-empty subset of S. Then there exists a homeomorphism of S with A as its fixed point set.*

PROOF. It is clearly sufficient to prove the theorem for the normal forms of compact surfaces: the 2-sphere, connected sums of tori, and connected sums of projective planes. For the 2-sphere, the theorem follows from Theorem 3 below

(i) Let S be the connected sum of m tori ($m \geqq 1$). We modify the standard construction of S from a fundamental polygon and obtain S as a quotient space of the rectangle $I = \{(x,y) \,|\, 0 \leqq x \leqq 2m,\ 0 \leqq y \leqq 1\}$ in the way indicated in Fig. 1. Hence the quotient map $q: I \to S$ identifies the subset

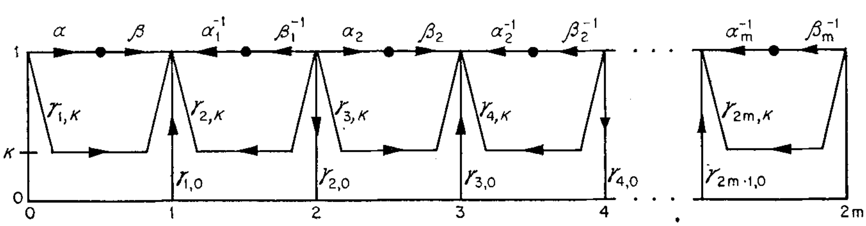

Fig. 1

$J = \{(x,y) \,|\, x = 0,\ x = 2m,\ y = 0,\ \text{or}\ x = \frac{1}{2}, \frac{4}{2}, \cdots, (4m-1)/2 \text{ and } y = 1\}$ of I to a point $a \in S$, and identifies two corresponding points of the oriented segments α_i, α_i^{-1} respectively β_i, β_i^{-1} $(i = 1, 2, \cdots, m)$.

Next define a family of arcs $\gamma_{1,k}$ $(0 < k < 1)$ on

$$I_1 = \{(x,y) \,|\, 0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1\}$$

by $\gamma_{1\,k} = \{(x,y) \,|\, y = y_{1,k}(x)\}$, where

$$y_{1,k}(x) = \begin{cases} \dfrac{2(k-1)}{k}x + 1, & 0 \leqq x < k/2, \\ k & k/2 \leqq x \leqq 1 - k/2, \\ -\dfrac{2(k-1)}{k}x + 3 - 2/k, & 1 - k/2 < x \leqq 1. \end{cases}$$

Reflect I_1 on $x = 1$ to obtain a family of arcs $\gamma_{2,k}$ on

$$I_2 = \{(x,y) \,|\, 1 \leqq x \leqq 2,\ 0 \leqq y \leqq 1\},$$

then reflect I_2 on $x = 2$ to obtain a family $\gamma_{3,k}$, and continue until families $\gamma_{j,k}$ for all $j = 1,2,\cdots,2m$ are constructed. Further define $\gamma_{j,0} = \{(x,y) \,|\, x = j,\ 0 \leqq y \leqq 1\}$ for $j = 1,2,\cdots,2m-1$.

Now orient the arcs $\gamma_{j,k}$ so that $\gamma_{j,k}$ starts at the point $(2[j/2], 1)$ if $j = 1,2,\cdots,2m$ and $0 < k < 1$; that $\gamma_{j,0}$ starts at $(j,0)$ if j is odd and at $(j,1)$ if j is even. Then the quotient map $q\colon (I,J) \to (S,a)$ will define, as the images of the oriented arcs α_i, β_i, and $\gamma_{j,k}$, a continuous family Φ of curves on S which are all homeomorphic to circles, and are mutually disjoint apart from the point a which is common to all of them. As there exists a homeomorphism of S onto itself which transforms a into any other given point of S we can assume that $a \in A$.

Choose a metric d for S compatible with the quotient topology. For every $p \in S \setminus \{a\}$ let ϕ_p be the member of Φ for which $p \in \phi_p$. If $p', p'' \in \phi_p$, denote by $s(p',p'')$ the length of the arc from p' to p'' along ϕ_p, where the direction of the arc is taken such that it coincides with the orientation of ϕ_p. For every $p \in S \setminus \{a\}$ define $g(p)$ as the point on ϕ_p for which p is between a and $g(p)$ in the orientation of ϕ_p, and for which

$$s(p, g(p)) = \tfrac{1}{2}d(p, A).$$

As $d(p,A) \leqq d(p,a) \leqq s(p,a)$, such a point always exists. Put further $g(a) = a$. Then g is a continuous self-map of S with A as its fixed point set.

The map g is an injection: If $p \neq a$, then $g(p) \neq a$, as otherwise $s(p,g(p)) = s(p,a) = \frac{1}{2}d(p,A)$ which is impossible. Hence $g(p') = g(p'')$ for $p', p'' \in S \setminus \{a\}$ implies that p' and p'' belong to the same curve ϕ_p of the family Φ, and that

$$s(a,p') + \tfrac{1}{2}d(p',A) = s(a,p'') + \tfrac{1}{2}d(p'',A).$$

We can assume that p' is between a and p''. Then

$$s(a,p'') - s(a,p') = s(p',p'') \geqq d(p',p'').$$

But on the other hand

$$s(a,p'') - s(a,p') = \tfrac{1}{2}(d(p',A) - d(p'',A)) \leqq \tfrac{1}{2}d(p',p''),$$

hence $p' = p''$.

The map g is also a surjection: A homotopy between g and the identity map of S is defined by the condition that $g(p,t)$, for $p \in S \setminus \{a\}$ and $0 \leqq t \leqq 1$, is the point on ϕ_p such that

$$s(a, g(p,t)) = s(a,p) + (1-t)\, s(p, g(p)),$$

and $g(a,t) = a$ for $0 \leqq t \leqq 1$. Hence $g_*: H_2(S) \to H_2(S)$ is the identity isomorphism of the second homology group of S with integer coefficients. Therefore g_* cannot be factored through $H_2(S \setminus \{q\}) = 0$ for any $q \in S$, so that the image of S under g cannot be contained in $S \setminus \{q\}$.

Hence g is the desired homeomorphism.

(ii) Now let S be the connected sum of m projective planes ($m \geqq 1$). The construction of a homeomorphism in this case is similar to the one just described. This time S is obtained from

$$I = \{(x,y) \,|\, 0 \leqq x \leqq 2m,\ 0 \leqq y \leqq 1\}$$

by identifying

$$J = \{(x,y) \,|\, x = 0,\ x = 2m,\ y = 0, \text{ or } x = 1,2,\cdots,2m-1 \text{ and } y = 1\}$$

to a point $a \in S$, and identifying every two points with coordinates $(2i + r, 1)$ and $(2i + r + 1, 1)$ for $0 < r < 1$ and $i = 0,1,\cdots,m-1$. As a family of arcs on I we choose the line segments of length $2m$ and parallel to the x-axis for $0 \leqq y < 1$, as well as the segments from $(j,1)$ to $(j+1,1)$ on $y = 1$ for $j = 0,1,\cdots,2m-1$, and orient them from left to right. The quotient map $q: (I,J) \to (S,a)$ then defines again on S a family of oriented curves for which a construction completely analogous to (i) yields an injection g of S with A as fixed point set. To prove that g is surjective, use twisted coefficients.

This completes the proof of Theorem 1.

3. Fixed point free homeomorphisms of surfaces

We assumed in §2 that the fixed point set A of the homeomorphism of the surface S should be non-empty. This is clearly necessary if S is homeomorphic to the projective plane and hence has the fixed point property. But in all other cases there exist fixed point free homeomorphisms, as the next theorem shows.

THEOREM 2. *Let S be a compact surface which is not homeomorphic to the projective plane. Then S admits a fixed point free homeomorphism.*

PROOF. Again it is clearly sufficient to prove the theorem for the normal types of compact surfaces. For the sphere and the torus, a fixed point free homeomorphism is e.g., the reflection on the centre of symmetry of the surface.

(i) If S is the connected sum of m tori and $m > 1$, then S can be represented as a sphere which has m handles attached to it along the equator in such a way that S is transformed onto itself by a rotation r_1 around the north-south axis through $2\pi/m$ degrees as well as by a reflection r_2 on the equatorial plane. Now define a fixed point free homeomorphism h by $h = r_2 \circ r_1$.

(ii) If S is the connected sum of m projective planes and $m > 1$, then S can be obtained by attaching m Möbius strips to the boundaries of m circular holes of a 2-sphere. Let S_0 be a 2-sphere from which n open discs have been removed in such a way that S_0 is transformed onto itself by homeomorphisms r_1 and r_2 defined as in (i). Attach a Möbius strip to each hole to obtain S. It is clearly possible to extend r_1 and r_2 from homeomorphisms of S_0 to homemorphisms of S and to define a fixed point free homeomorphism of S as their product.

4. Some results for compact n-manifolds

We first extend theorems 1 and 2 to n-spheres for $n > 2$, and fill the gaps left in the proof of theorem 1.

THEOREM 3. *Let A be an arbitrary closed subset of an n-sphere $S^n (n \geqq 1)$. Then S^n admits a homeomorphism with A as its fixed point set.*

PROOF. If $A \neq \emptyset$, then a homeomorphism of an $(n + 1)$-ball constructed in the proof of th. 1 in [3] yields a suitable homeomorphism by restriction to the boundary of the ball. If $A = \emptyset$, then the antipodal map will do.

We finally prove a weaker form of Theorems 1 and 2 for compact n-manifolds, in which homeomorphisms have been replaced by surjections. By an n-manifold we mean a connected separable metric space M such that every point p has a neighbourhood homeomorphic to Euclidean n-space. We use the concept of a path field on M [1,2], which is a map $f: M \to M^I$ such that $f(p)$ is either a path $\sigma(p,t)$, with $0 \leqq t \leqq 1$ and $\sigma(p,t) = p$ if and only if $t = 0$, or the constant path at p. It was proved by R. F. Brown [1, th. 1.11] that every compact n-manifold admits a path field which has at most one constant path.

THEOREM 4. *Let A be an arbitrarily closed subset of a compact n-manifold which is non-empty if the Euler characteristic of M is non-zero. Then M admits a surjective self-map with A as its fixed point set.*

PROOF. Consider first the case $A \neq \emptyset$, and let $f: M \to M^I$ be a path field on M which has at most one constant path. If the path field contains one constant path, we can assume that it occurs at a point $a \in A$. Let d be a metric of M and take $d'(p) = \min(1, d(p, A))$ for $p \in M$. Define $g: M \to M$ by

$$g(p) = \begin{cases} \sigma(p, d'(p)) & \text{if } p \neq a, \\ a & \text{if } p = a. \end{cases}$$

Then g is a continuous self-map of M with fixed point set A. An argument analogous to the one used in the proof of Theorem 1 shows that g is surjective.

That M admits a fixed point free surjection if its Euler characteristic is zero follows from [2, corol. 4.6].

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